THE SPRING–MASS MODEL FOR RUNNING AND HOPPING

R. BLICKHAN*

Concord Field Station, Harvard University, Bedford, MA 01730, U.S.A.

Abstract—A simple spring–mass model consisting of a massless spring attached to a point mass describes the interdependency of mechanical parameters characterizing running and hopping of humans as a function of speed. The bouncing mechanism itself results in a confinement of the free parameter space where solutions can be found. In particular, bouncing frequency and vertical displacement are closely related. Only a few parameters, such as the vector of the specific landing velocity and the specific leg length, are sufficient to determine the point of operation of the system. There are more physiological constraints than independent parameters. As constraints limit the parameter space where hopping is possible, they must be tuned to each other in order to allow for hopping at all. Within the range of physiologically possible hopping frequencies, a human hopper selects a frequency where the largest amount of energy can be delivered and still be stored elastically. During running and hopping animals use flat angles of the landing velocity resulting in maximum contact length. In this situation ground reaction force is proportional to specific contact time and total displacement is proportional to the square of the step duration. Contact time and hopping frequency are not simply determined by the natural frequency of the spring-mass system, but are influenced largely by the vector of the landing velocity. Differences in the aerial phase or in the angle of the landing velocity result in the different kinematic and dynamic patterns observed during running and hopping. Despite these differences, the model predicts the mass specific energy fluctuations of the center of mass per distance to be similar for runners and hoppers and similar to empirical data obtained for animals of various size.

NOMENCLATURE

m	mass
g	gravitational acceleration
k	stiffness
l	spring length
ω	natural frequency of the spring–mass system
x	horizontal deflection
y	vertical deflection
y_a	hopping height
y_c	contact displacement
$\dot{y}_a$	vertical landing or take off velocity
t_a	aerial time
t_c	contact time
T	period of a bounce
f	frequency of a bounce
F	force
F_{peak}	(vertical) peak force
$F_{x,y}$	peak horizontal and vertical forces
α	angle of attack of the spring
β	angle of the landing (take off) velocity
$\Delta l_{x,y}$	changes in $l_{x,y}$
suffix s	specific
x, y	horizontal and vertical accelerations

INTRODUCTION

Animals use a bouncing gait during rapid terrestrial locomotion. This was established from forceplate measurements for a large number of animals (Cavagna *et al.*, 1977; Heglund *et al.*, 1982). Bouncing is a measure to minimize energy expenditure similar

to the way the interchange between potential gravitational and kinetic energy saves energy during slow locomotion. Some animals can store up to 70% of the kinetic energy delivered during landing in elastic tissues which can then be used for take off (Alexander and Vernon, 1975).

Several recent findings suggest that the analogy of hopping to a simple spring–mass system may be appropriate. During hopping or galloping animals prefer a particular stride frequency (Heglund and Taylor, 1988). Experiments on human hopping have shown that it is indeed more expensive to drive the system at a frequency below this preferred frequency (Perez, personal communication) as would be expected for a spring–mass system driven at frequencies different from the resonant frequency. However, if the whole body stiffness is variable then the natural frequency can be changed and, therefore, other factors might determine preferred frequency.

The animal's muscle skeletal system can be considered mechanically as an actively-driven, nonlinear, multicomponent spring–mass system. Here it is assumed that it behaves simply like a point mass bouncing passively on a massless spring without viscous losses. This is the simplest model possible for any bouncing system. The advantage of such a simplifying approach is its transparency with respect to the influence of physical and morphological conditions. The model offers important insights and describes the interdependency of the parameters characterizing running and hopping.

Starting with the simplest case, hopping in place, the interdependency of the aerial and contact phase is

described. The subdivision between these two phases, or the hopping pattern, can be characterized by several variables, including the specific vertical force. From three given parameters, such as body mass, hopping frequency, and contact time, all remaining parameters characterizing human hopping in place can be estimated. Physiological constraints result in a narrow frequency region where hopping is possible. The preferred hopping frequency is the most economical one.

With the introduction of forward speed the state of the spring–mass system is characterized by five independent variables and the length of the spring and centrifugal forces become relevant. The model predicts that contact length becomes maximal for the flat landing angles chosen by animals during running and hopping. For landing angles between 5 and 25 degrees the peak vertical ground reaction force and the total vertical displacement can be predicted from contact time and step frequency. The vertical displacements during running are lower than those during hopping, i.e. running results in a smoother ride. Nevertheless, the model predicts that the sum of the energy fluctuations over a given distance is similar for runners and for hoppers, in agreement with measurements carried out on animals of various size.

HOPPING AT ZERO SPEED

The seemingly artificial situation of hopping in place, i.e. at zero speed, can be taken as a model for bouncing gaits in animals (Farley et al., 1985). Even during hopping in place a man prefers the same hopping frequency as used by a kangaroo of similar size and the same frequency that an antelope of similar bodymass prefers as stride frequency during galloping.

The simplest hopping model possible, a one-dimensional spring–mass system (Fig. 1), correctly predicts the interdependence and magnitude of all major mechanical parameters.

Description of the spring–mass model

The mathematics of a simple spring–mass system defines the relevant variables and points to the assumptions involved.

Assuming a linear spring (i.e. deflection proportional to applied force) the following equation describes the motion of the mass during ground contact:

$$m\ddot{y} + ky = mg, \tag{1}$$

where m is body mass, y is vertical deflection, k is stiffness and g is gravitational acceleration.

The general solution of this equation consists of a linear combination of the solution of the homogeneous differential equation and a special solution of the inhomogeneous equation:

$$y = a\sin\omega t + b\cos\omega t + g/\omega^2, \tag{2}$$

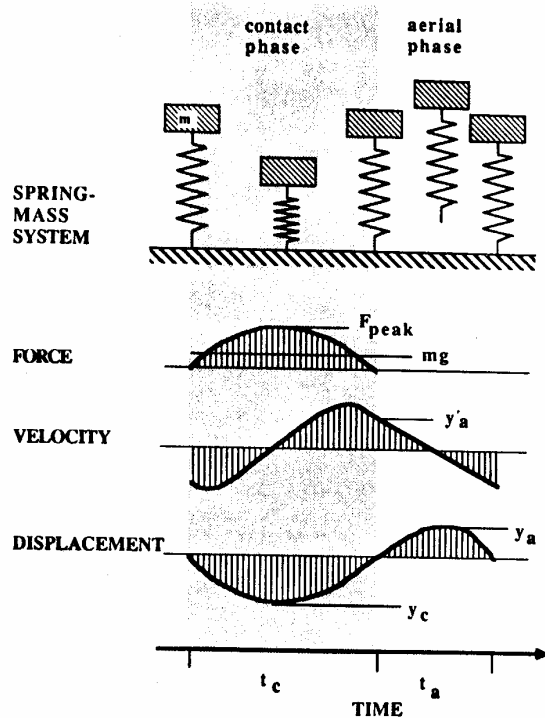with $\omega^2 = k/m$; ω is natural frequency.

Fig. 1. Spring–mass model, a point mass attached to a massless spring. The hopping period (T) is subdivided into two phases, a contact phase (t_c) and an aerial phase (t_a). A sinusoidal displacement during contact time is followed by a parabolic flight phase. Due to the linear stiffness of the spring the time course of the force is also sinusoidal with the weight of the system as equilibrium value. The length of the flight phase determines the landing or take off velocity ($\dot{y}_a$). The maximum velocity is reached after landing when the force equals body weight (t_c: contact time).

The constants a and b are determined by boundary conditions. A necessary condition for hopping is that the spring force ($F \propto y$) exerted at the moment of touch-down ($t = 0$) is 0:

$$y(t = 0) = 0, \quad \text{thus} \quad b = -g/\omega^2.$$

The remaining factor (a) can be related to the landing velocity ($\dot{y}_a$):

$$\dot{y}(t = 0) = \dot{y}_a, \quad \text{thus} \quad a = \dot{y}_a/\omega.$$

The resulting solution is:

$$y = \dot{y}_a/\omega \sin\omega t - g/\omega^2\cos\omega t + g/\omega^2 \tag{3}$$

or

$$y = \dot{y}_a(m/k)^{1/2}\sin\omega t - gm/k\cos\omega t + gm/k. \tag{3a}$$

From this, the force during stance (F) can be calculated as:

$$F = \dot{y}_a(km)^{1/2}\sin\omega t - gm\cos\omega t + gm. \tag{3b}$$

At midstance ($t = t_c/2$; t_c is contact time), the spring is deflected to its maximum and the velocity of the center of mass is zero ($\dot{y}(t = t_c/2) = 0$):

$$\dot{y}_a\cos(\omega t_c/2) + g/w\sin(\omega t_c/2) = 0,$$

thus $$\tan(\omega t_c/2) = -\dot{y}_a\omega/g. \tag{4}$$

This nonlinear equation relates take off or landing velocity, contact time, and the natural frequency of the spring–mass system. Selection of values for two of these three parameters uniquely determines the third one. Respective numbers can be calculated in a numerical, iterative approximation.

During the *aerial phase* the equation of motion is simply:

$$m\ddot{y} = mg, \quad \text{thus} \quad \dot{y}_a = g t_a/2, \quad (5)$$

where t_a is aerial time.

The contact phase can be characterized by the landing velocity $(\dot{y}_a)$ and the natural frequency of the oscillator (ω), whilst the aerial phase depends only on the landing velocity $(\dot{y}_a)$. Thus the aerial phase depends unambiguously on the stance phase or vice versa. The *stride period* (T) and the hopping frequency $(f = 1/T)$ are determined by:

$$T = t_c + t_a. \quad (6)$$

Of all mechanical variables describing steady state hopping at zero speed $(f, T, t_c, t_a, y, y_c, y_a, F_{peak}, k, \omega, \dot{y}_a, \text{etc.})$, only three are independent. The performance of the spring–mass system depends on the magnitude of these selected parameters.

The hopping pattern

Despite very different frequencies and amplitudes, hopping looks similar for a small kangaroo rat and a large red kangaroo: hops have similar "patterns". This similarity in pattern seems to be characterized by a certain ratio of contact time (t_c) and aerial time (t_a), by a certain ratio of the displacements during ground contact (y_c) and during the aerial phase (y_a), or by a certain time course of the ground reaction force $(F(t))$. It is worthwhile to describe this pattern and to understand its variation because the pattern typical of animals differs much from a bouncing steel ball generally taken as the typical example of a system bouncing elastically.

Dividing force by body weight results in a dimensionless formulation of equation (3b):

$$F_s(t) = \dot{y}_{as}\sin\omega t + 1 - \cos\omega t \quad (3c)$$

where $F_s = F/mg$ represents the specific ground reaction force which is equal to the specific displacement during ground contact $(y_{cs} = y_c\omega^2/g)$, and $\dot{y}_a\omega/g$ the specific landing velocity, which is proportional to the specific aerial time $(t_{as} = t_a\omega = 2\dot{y}_{as})$, and proportional to the square root of the aerial displacement $(y_{as} = y_a w^2/g = \dot{y}_{as}^2/2)$. From equation (4) we know that a given specific landing velocity corresponds to a certain specific contact time $(t_{cs} = t_c\omega/2)$. Thus, the specific landing velocity also determines the ratio of contact and aerial time. Rewriting equation (3) using the specific displacements shows that the specific landing velocity also determines the ratio of the amplitudes. All dimensionless ratios can be used for characterization of the hopping pattern.

The specific landing velocity $\dot{y}_{as}$ or its equivalents determine the amplitude and the time course of the specific ground reaction force, i.e. the partitioning of the sine or cosine terms. For *small specific aerial times* the sine term vanishes. The mass oscillates around the resting position determined by the spring stiffness and the animal's weight. The contact time approaches the period of the oscillator. For *large specific aerial times* the sine term dominates. The time course of the ground reaction force approaches a sine half-wave whose amplitude greatly exceeds the animal's weight. The contact time becomes small compared to the time the animal spends in the air. Hopping then consists of a sequence of parabolic flight phases and the system bounces very much like a steel ball on a hard surface.

The influence of constraints

Due to anatomical and physiological properties of their locomotory system animals can use only a very limited region of the parameter space theoretically possible. For example, it is impossible for a human hopper to hop with a frequency higher than 10 Hz, or to hop to a height of 3 m, or to hop like a steel ball.

Including body mass *three* parameters must be given in order to set the point of operation of a spring–mass system hopping in place. However, due to the interdependency of all parameters it is not sufficient to consider constraints in only three parameters. The constraints of one parameter should not conflict with constraints of another. In the case of conflicting constraints hopping would become physically impossible.

How do physiological constraints cause a narrow range of *possible hopping frequencies* (Fig. 2)?

Force (F_{peak}). The isometric stress developed by vertebrate muscles is about 250 kPa (Close, 1972). From this a maximum ground reaction force of *ca* 3 kN can be estimated for an average human hopper (70 kg). This limit in force implies that the contact time cannot become very small compared to the duration of the top. In order to keep the forces below a certain limit at low frequencies spring stiffness must be low. In contrast, high stiffnesses are required to be able to generate high forces at high frequencies.

Contact displacement (y_c). The maximum displacement during ground contact cannot exceed the leg length which is about 1 m for a human hopper. If ground reaction force should be close to maximum this length shrinks to about 0.5 m due to the force–length curve of the muscles and the increase of the moment arm with knee flection. The assumption that during hopping muscles work largely isometrically and the deflection is taken up by lengthening of the tendons would reduce this number further.

For low frequencies limited contact displacements invoke high whole body stiffnesses and high ground reaction forces. The limits in force and displacement result in a lower boundary for the hopping frequency.

RESTRICTED PARAMETERS
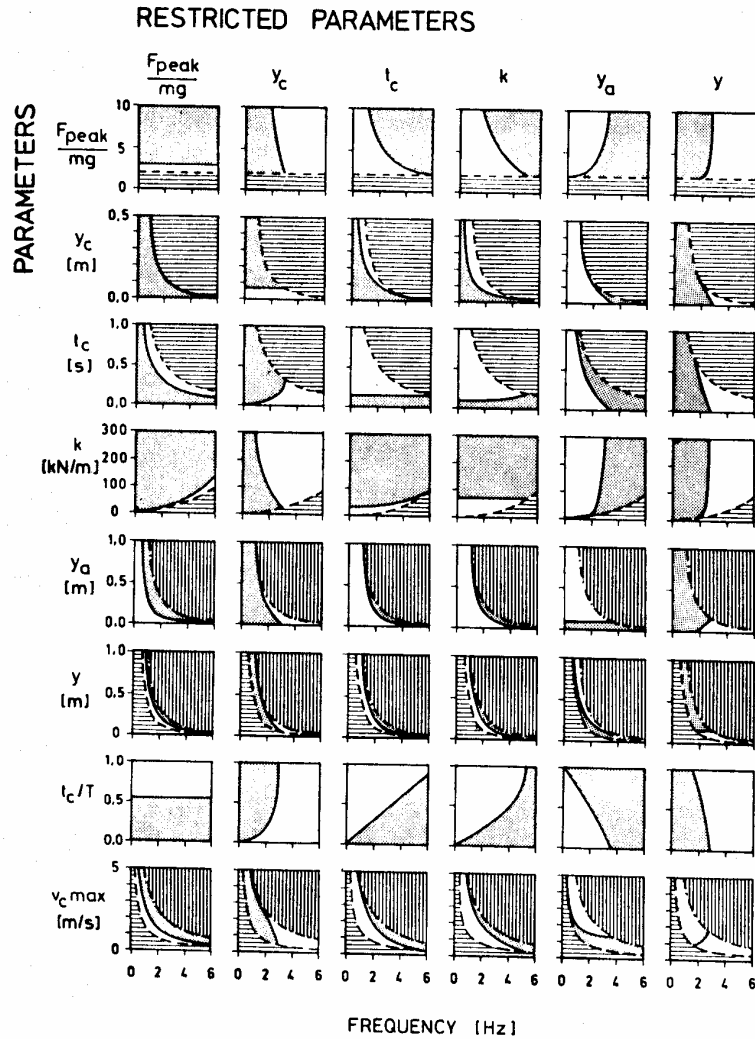

FREQUENCY [Hz]

Fig. 2. Influence of constraints on a vertically bouncing spring-mass system (mass: 64 kg). Without any restriction the bouncing mechanism itself imposes limits on most parameters (vertical hatching: $t_c = 0$ or $t_a = T \Rightarrow F_{ypeak}/mg = \infty, y_c = 0, k = \infty, y_a = g/(8f^2), y = y_a, t_c/T = 0, v_{cmax} = g/2f$; horizontal hatching: $t_c = T, t_a = 0 \Rightarrow F_{peak}/mg = 2, y_c = g/(2\pi^2 f^2), k = m4\pi^2 f^2, v_{cmax} = g/(2\pi f)$; unhatched: parameter space where bouncing is possible). Stippled regions: areas which are excluded after introducing constraints (from left column to right): specific force $F/mg < 3$; contact displacement $y_c < 0.06$ m; contact time $t_c > 0.15$ s; spring stiffness $k < 70$ kN m^{-1}; aerial displacement $y_a > 0.1$ m; total displacement $y < 0.15$ m. ($T = f^{-1}$; v_{cmax}: maximum shortening velocity of the spring.)

If the feedback loop controlling muscle forces requires a particular minimum displacement, then this displacement limits the maximum hopping frequency.

Stiffness (k). Maximum energy can be delivered for high forces and high displacements. The values given above for the maximum forces and displacements result in a corresponding stiffness value of about 6 kN m^{-1}. In terms of energy storage less force, i.e. less muscle tissue, is necessary to store the same amount of energy in a more compliant system. It is thus not economical to build very stiff systems.

In a mammalian leg at least one limb segment is oriented obliquely to the direction of the load. As a result, attainable stiffness values are largely determined by the properties of muscles, tendons and ligaments. High stiffness values can be attained by locking the more compliant joint. If we assume that a human hopper hops with his knees locked then the

properties of the achilles tendon should determine maximum stiffness. A rough estimation (Young's modulus: 1.2 GN m^{-2}; tendon length: ca 3 m; tendon cross-sectional area: ca 50 mm^2) yields a tendon stiffness of ca 200 kN m^{-1}. High stiffness of a joint can be attained by keeping it as straight as possible during landing. Assuming a lever arm ratio of 2:1 the stiffness of one leg could reach 100 kN m^{-1}.

The stiffness along with the body mass determines the natural frequency of the oscillator and it limits the range of possible contact times from one half of the natural period of the spring-mass system for low frequencies to the full natural period for the highest frequencies.

Contact time (t$_c$). If the joints are not locked all changes of the ground reaction forces are accompanied by equivalent changes in muscle force. The intrinsic time constants of the muscles involved would

introduce upper and lower boundaries for contact times. If an animal uses prestretch to enhance the force output the stretching has to occur rapidly compared to relaxation processes triggered simultaneously. The shortest contact time is determined by the fastest switching time of all muscles involved. Contact times measured for humans hopping between 2 and 6 Hz range from 0.1 to 0.3 s (Farley et al., 1985).

For a limited contact time lower frequencies can only be reached by increasing aerial time, i.e. the landing velocity. This in turn requires high forces. Considering the fact that short contact times require high intrinsic muscle velocities and are thus metabolically expensive, these contact times should be adapted to the largest ground reaction forces which can be exerted by the leg. Moreover, the velocity of the center of mass increases with decreasing hopping frequency. In other words it is not necessary to decrease contact time if the forces are too low to be able to generate the momentum for take off during that time. The highest hopping frequency is attained when the hopping frequency equals the inverse of the shortest contact time possible. A constraint in the contact time imposes a limit to the stiffness values and vice versa.

Aerial displacement (y_a). Terrestrial habitats possess a certain surface roughness which sets a minimum

hopping height. A minimum aerial time necessary for the recovery stroke would also result in a lower limit for the hopping height. The maximum height could be selected freely. It is, however, obvious that within a limited time span high jumps can only be provided by sufficient high ground reaction forces and stiffnesses.

Total displacement (y). In experiments on humans the total displacement during each hop can be controlled more easily than any other parameter discussed so far (Farley et al., 1985). As the spring–mass model demonstrates, the condition of a constant total displacement results in an extremely narrow range of possible hopping frequencies. The hopping frequency could thus be approximately predicted from the total displacement.

The hopping frequency. Combining the rough constraints results in the frequency (1–6 Hz) and pattern $(t_c/T > 0.3)$ region observed for human hoppers (Fig. 3). Certainly, the constraints are tuned to each other. They are not conflicting, i.e. hopping is still possible, and they overlap to a large degree.

Human hopping at maximum height: comparison between model and experiment

Hopping in place at maximum height over a wide range of hopping frequencies (1–6 hops s^{-1}) provides (a) the range of possible hopping frequencies for a
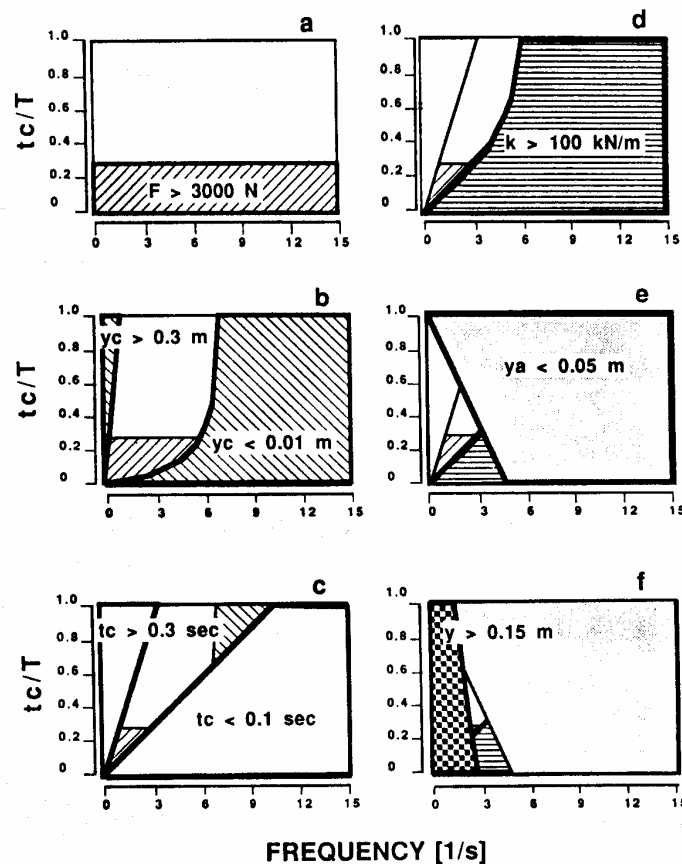
Fig. 3. Range (unshaded) of hopping frequencies and hopping patterns (ordinate), measured by t_c/T, possible after introducing physiologically reasonable constraints (shaded: parameter space forbidden by the constraint). The constraints are introduced step by step, (a) F; (b) y_c; (c) t_c; (d) k; (e) y_a; (f) y, with the new forbidden regions lined out heavily.

human hopper, (b) the maximum output of the system at these frequencies (Farley et al., 1985), and (c) a direct comparison between model and experiment (Blickhan et al., 1986).

Humans can set hopping frequency to a metronome. For an individual hopper the body mass is given leaving one parameter free. This is set according to individual abilities under the condition of maximum hopping height. Forceplate measurements allow for the calculation of the force, contact time, and hopping frequency and can be used to calculate velocities and displacements (Cavagna et al., 1976). Under the condition of maximum height the contact time remains almost constant in the frequency range from 2 to 6 hops s^{-1}, where ca 2 Hz is the preferred frequency (Fig. 4; Farley et al., 1985). Based on this, contact time, peak ground reaction force, peak contact displcement and whole body stiffness are calculated using the spring–mass model. The deviation of the predicted values from the measured ones is less than 20%.

The predictions for hopping in place seem not to be sensitive to minor deviations from the assumptions used in the calculations and they do not depend on finding a linear elastic spring in the musculo-skeletal system. It is sufficient that the control of the musculo-skeletal system results in a nearly spring-like behavior during ground contact.

Interpretation of the data on human hopping

The data show that during hopping at maximum height stiffness increases with the hopping frequency. This increase of stiffness (k) is, however, not sufficient to maintain the same pattern: the ratios of aerial time to contact time as well as the peak ground reaction forces are changing. A constant pattern would require an increase of the stiffness with the square of the hopping frequency. In fact, from the constant contact time it can be deduced that the stiffness can vary up to a factor of four over the whole frequency range.

The maximum stiffness attainable, ca 90 kN m^{-1} for an untrained human hopper, sets the maximum hopping frequency at about 6 hops s^{-1}. This stiffness value is about half the value assessed from anthropomorphic data assuming only rotation of the ankle joint. In a multi-joint system stiffness can be increased by locking one joint. A closer look at the kinematics of the human leg (Farley et al., 1985) demonstrates that the measured 90 kN m^{-1} for whole body stiffness may well be an overestimation if applied to the ankle joint. The knee is not simply locked but moves ca 180° out of phase. It acts as if it would drive the ankle joint actively inducing higher deflections than measured for the center of mass. In this case, both the leg mass and the stiffness of the ankle joint determine the contact time. Thus, the maximum possible ankle stiffness may well be half of the measured whole body stiffness. The change in stiffness with increasing frequencies is largely induced by leg kinematics and not by material properties.
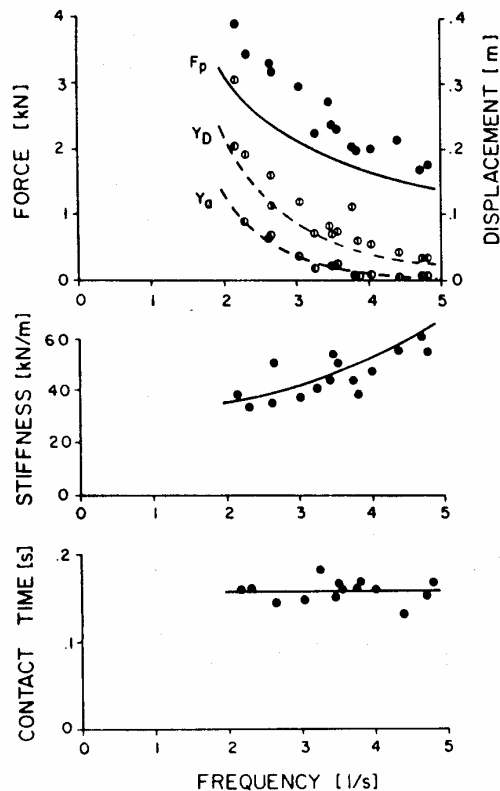

Fig. 4. Hopping in place at maximum height: comparison of measurements and predictions. The contact time is fairly constant (c) as shown by the fitted line. (a) Peak forces (closed circles) and peak displacements (total displacement: circle with slash; hopping height: circles half filled) are slightly larger than those predicted from the contact time using the spring–mass model. These errors cancel each other out in the calculation of the whole body stiffness.

Below the preferred frequency of ca 2 Hz the force pattern changes dramatically (Taylor, 1985). The occurrence of multiple peaks in the force tracings indicate that a simple spring–mass model might be of very limited value. Apparently the hopper avoids large excursions of the center of mass by changing the force pattern. Compared to a sinusoidal force pattern, a more rectangular pattern allows application of the same momentum with reduced peak displacement and reduced force during ground contact.

Why is it impossible to use locking of the knee to increase stiffness at low frequencies? Hopping with the same stiffness at lower frequencies would demand much higher displacements during ground contact which in turn cannot be supplied by the stiffened leg: the knee has to be flexed. Using the knee as a part of the spring automatically reduces the spring's stiffness, but increases the range of possible deflections. The lower stiffness value does not result in longer contact times. The more compliant spring even stores a higher amount of energy in the same time if loaded with a higher momentum which is possible by hopping at a lower frequency. Thus, the preferred frequency allows for the development of the highest forces and the

delivery and storage of large amounts of mechanical energy.

If we are not interested in maximum height, hopping frequencies from ca 1–6 Hz can be chosen. Why then should a human hopper still have an invariant frequency? The high stiffnesses during hopping at high frequencies demand high muscle forces without gaining the benefit of high jumps. Even for the limit of an infinitely high stiffness the jumping height can only reach ca 0.035 m for a hopping frequency of 6 hops s^{-1}. Hopping using the preferred frequency allows a much higher jump. If maximum height is not required stiffness can be reduced allowing for longer contact times. This in turn reduces muscle recruitment and lowers switching and loading rates. As long as elastic storage is possible, i.e. the deflections are small enough and the loading rates are not too slow, it is more economical to hop at a low frequency.

HOPPING FORWARD

For hopping in place theoretical predictions and empirical data are in agreement. Next, planar movement, i.e. bouncing at a given forward speed, is considered. In the one-dimensional model formulated above the geometric dimensions of the system merely limit possible excursions during ground contact. However, hopping at nonzero speed introduces leg length as an additional parameter in the governing differential equations of the spring–mass system. Contact time not only depends on the spring stiffness and the vertical landing velocity, but also on the animal's speed and leg length. Despite the enlarged parameter space, a simple spring–mass model is still powerful in describing the performance of running or hopping animals. In particular, it will be demonstrated how the mechanical properties of a simple spring–mass system working under physiological constraints correctly predicts the mass specific sum of the fluctuations of the energy of the center of mass at a given distance. This parameter was found to be the same for a large variety of animals (Heglund et al., 1982; Blickhan and Full, 1987).

The spring–mass system bouncing forward

During ground contact the planar movement of a point-like mass attached to a massless spring (Fig. 5a) can be described by a system of two nonlinear differential equations:

$$\ddot{x} = x\omega^2(l/(x^2 + y^2)^{1/2} - 1) \tag{7a}$$

$$\ddot{y} = y\omega^2(l/(x^2 + y^2)^{1/2} - 1) - g \tag{7b}$$

where x = horizontal deflection, y = vertical deflection, g = gravitational acceleration, $\omega = (k/m)^{1/2}$ = natural frequency of a spring–mass system, k = spring stiffness, m = mass, l = length of the spring $= (x_0^2 + y_0^2)^{1/2}$.

The inclusion of the horizontal component results in the appearance of the spring length (l) in these
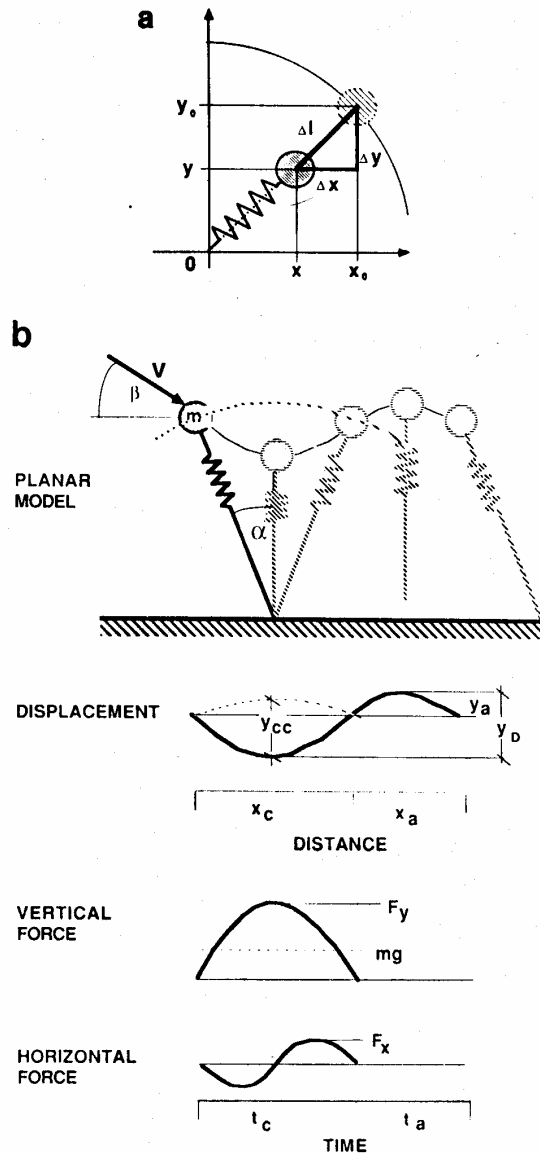
Fig. 5. The planar model. Here force, velocity, and displacement become vector quantities. (a) The compression of the spring (Δl) can be subdivided in a horizontal (Δx) and vertical (Δy) component. (b) Of all simulated hops only stationary movements were selected, i.e. with symmetrical take off and landing velocities (v). l, length of the spring; v, absolute value of the landing velocity; β, angle of the landing velocity; α, angle of attack of the spring; x_c, contact length; x_a, hopping distance; y_{cc}, compression of the spring at midstance, not equal to the vertical displacement during ground contact y_c; $F_{y,x}$, peak vertical and horizontal force.

equations. For forward locomotion the length of the spring becomes as important as the spring stiffness and the landing velocity. The nonlinear coupling results from the fact that the forces, i.e. the accelerations, are *not* related linearly to the actual movement of the center of mass (x, y) but to the compression of the spring ($\Delta l = (\Delta x^2 + \Delta y^2)^{1/2}$; see Fig. 5a). Even for the case in which the center of mass does not show any vertical displacement and is only moving forward the compression of the spring shows a vertical component

(Δy) which exceeds the horizontal component (Δx). The coupling between the horizontal and vertical components implies that the fluctuations in the horizontal and vertical components of the energy of the center of mass (compare with Cavagna et al., 1977) do *not* translate to similar fluctuations of the horizontal and vertical work done on the spring.

The planar model has five independent parameters, e.g. the mass (m), the spring length (l), spring stiffness (k), the absolute value of the landing velocity (v), and the angle of the landing velocity (β). The system of equations was solved by using a fourth order Runge–Kutta algorithm (500 steps, step precision = 0.00001) on a PDP 11/23 laboratory computer. A cross-check of the algorithm by integrating the calculated displacements proved an agreement within 0.3% of the calculated take off velocities. Five parameters were given; one parameter, the angle of attack of the spring, was determined iteratively by taking the deviation of the take off velocity from the desired value as the guiding measure.

Pattern of running or hopping at nonzero speed

For hopping in place one parameter, e.g. the specific vertical landing velocity or the specific force, was sufficient to characterize the pattern of a hop. During

forward hopping the pattern also depends on the animal's leg length.

Again, this becomes more transparent in a dimensionless form of equations (7a,b):

$$\ddot{x}_s = x_s(l_s/(x_s^2 + y_s^2)^{1/2} - 1) \qquad (8a)$$

$$\ddot{y}_s = y_s(l_s/(x_s^2 + y_s^2)^{1/2} - 1) - 1 \qquad (8b)$$

where $x_s = x\omega^2/g =$ specific displacement, $\dot{x}_s = \dot{x}\omega/g =$ sp. velocity, $\ddot{x}_s = \ddot{x}/g =$ sp. acceleration and $t_s = t\omega =$ sp. time.

Constants. Specific vertical landing velocity: $\dot{y}_{as} = \dot{y}_a\omega/g$; specific spring length: $l_s = l\omega^2/g$.

The motion of the mass characterized by equations (8a,b) can only be similar if the specific landing velocity *and* the specific spring length are the same. The specific length (l_s) is the same if the body's weight (mg) compresses the spring by the same relative length ($\Delta l_{weight}/l$):

$$l_s = l\omega^2/g = lk/mg = l/\Delta l_{weight}, \qquad (9)$$

where k is the spring stiffness.

Interdependence of parameters

Even if we only consider animals of a certain size, i.e. if we keep mass and spring length constant, an infinite variety of hopping patterns are still possible
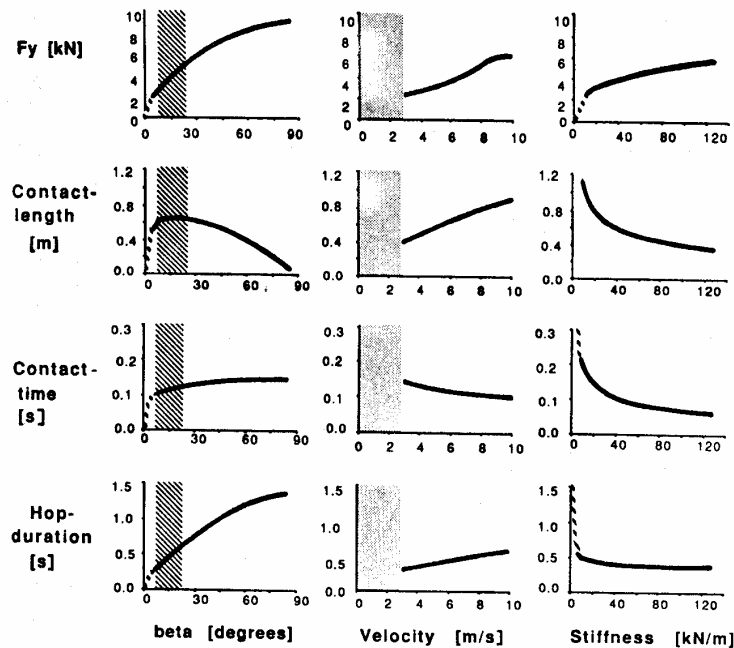

Fig. 6. Influence of constraints on planar hopping ($m = 64$ kg; $l = 0.9$ m). Column 1: ($k = 70$ kN m^{-1}; $v = 6$ m s^{-1}): animals use in general flat angles (hatched) of the landing velocity (β). For such angles the ground reaction force (F_y) becomes low and the contact length is at its maximum. For flat angles the contact time becomes shorter than half the natural frequency of the spring–mass system and the hopping frequency, the inverse of the hop duration, increases steeply. Column 2: ($k = 70$ kN m^{-1}; $\beta = 14°$): with increase of the landing velocity the ground reaction force increases. The stippled region marks the range of walking. For the selected point of operation the hopping frequency increases from 1.6 to 3.4 hops s^{-1}. The possible change in velocity does not influence the hopping frequency as much as the angle of the landing velocity (β). Column 3: ($v = 6$ m s^{-1}; $\beta = 14°$): in the relevant stiffness region ($k > 10$ kN m^{-1}) stiffness also has a minor influence on hopping frequency, whereas when the ground reaction force increases, the contact length and contact time are decreasing hyperbolically. The interrupted line represents an extrapolation; no solutions were found in this region using the numerical simulations.

by changing the vector of the landing velocity and the spring stiffness. As for hopping in place, physiological conditions limit the parameter space (Fig. 6). Here, only some important new aspects will be discussed.

Contact time and hop frequency both strongly depend on angle of the landing velocity to the horizontal (β). For very flat landing angles the hopping frequency increases due to short aerial times and short contact times induced by centrifugal forces. For very steep landing angles step length becomes short and the contact time and hop duration must decrease in order to achieve higher speeds. For a certain speed a maximum contact length was found for landing angles from 5 to 25 degrees. With speed a long contact length and large horizontal forces can only be maintained by a simultaneous increase in leg stiffness. This results in lower contact times and higher ground reaction forces.

Running and hopping animals indeed use angles of the landing velocity ranging from 5° to 25° (Blickhan, 1989). For such a restricted parameter space some parameters become closely related (Fig. 7).

(a) The *total vertical displacement* (y) and the *duration* (T) of the hop:

$$y \approx 1.1\, T^2. \tag{10}$$

(b) As during hopping in place the *specific vertical force* (F_y/mg) and the ratio between contact time and hop duration (t_c/T):

$$F_y/mg = y_c\omega^2/g \approx 1.7\, T/t_c. \tag{11}$$

(c) The ratio between *horizontal and vertical force* (F_x/F_y) and the spring's *angle of attack* (α):

$$F_x/F_y \approx 0.33 \tan\alpha. \tag{12}$$

These relationships allow the calculation of gait dynamics from kinematics or vice versa.

Human running or hopping

The planar spring–mass model describes well human running and hopping (Fig. 8).

Running. For a human runner the step frequency increases from *ca* 2.7 steps s^{-1} at a speed of 3 m s^{-1} to 4.2 steps s^{-1} at 9 m s^{-1}. Such an increase in step frequency is typical for all running and trotting animals (Heglund *et al.*, 1974). The distance a runner travels during ground contact, i.e. the contact length, is for all runners about equal to their leg length and varies little with the animal's speed (Cavagna *et al.*, 1977). Here, only solutions that meet these constraints are presented.

The stepping frequencies given for runners permit *angles* of the landing velocity between 5° and 10°. This produces a very *smooth ride*, i.e. small total vertical excursions. The restriction of the contact length induces an increase in spring *stiffness* with increasing running speed from *ca* 10 kN m^{-1} at 3 m s^{-1} to *ca* 30 kN m^{-1} at 9 m s^{-1}. These stiffness values are of correct magnitude (McMahon *et al.*, 1986) and result in shorter *contact times* at higher running speeds. The
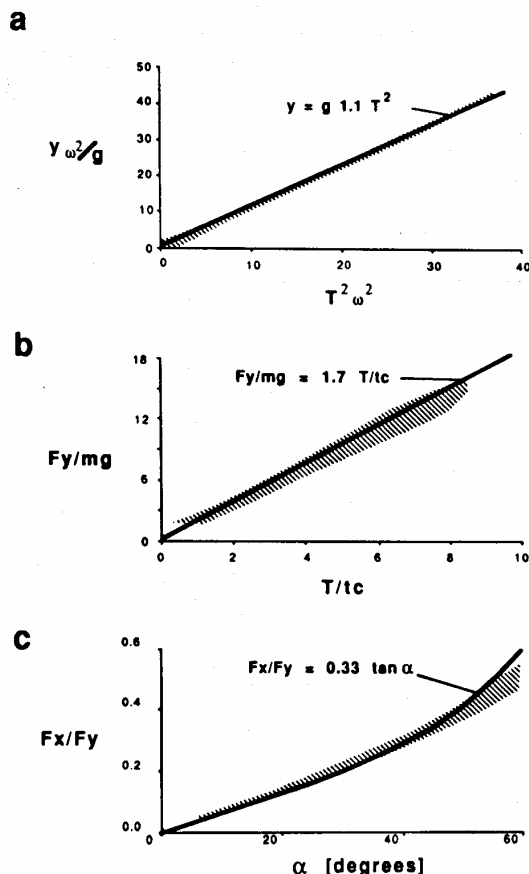
Fig. 7. Restriction to flat landing angles (5° ≤ β ≤ 25°). The stippled area marks the range where solutions were found. The lines are drawn by eyeball through these points. (a) The total vertical displacement (y) is proportional to T^2. (b) Specific force (F_y/mg) and the inverse of t_c/T are related linearly. (c) The ratio of the horizontal and vertical forces (F_x/F_y) is sinusoidally related to the angle of attack of the spring and thus proportional to the contact length.

calculated contact times range from 0.1 to 0.3 s (see Cavagna *et al.*, 1976). The peak *vertical force* increases with running speed attaining *ca* 3.5 kN at 9 m s^{-1}. This is about half the maximum force measured during bipedal hopping in place. A lower force and a longer contact time makes slow running more economical per unit time. The predicted ratio of the *horizontal and vertical force* (0.15–0.3) is within values measured for all animals (0.1–0.3; Cavagna *et al.*, 1976). The predicted specific vertical landing velocity ranges from 0.5 to 1.5. The model 'runs' with the same pattern as a human runner (McMahon *et al.*, 1986). Most significantly, the model predicts the energetics of the center of mass correctly (Cavagna *et al.*, 1976). The sum of all positive increments of the energy fluctuations of the center of mass over a given distance, the '*mechanical cost of transport*' amounts to *ca* 1.3 W kg^{-1} m^{-1} (cf. Blickhan and Full, 1987). The mechanics of the center of mass of an animal is the result of physiological constraints imposed on a bouncing system.
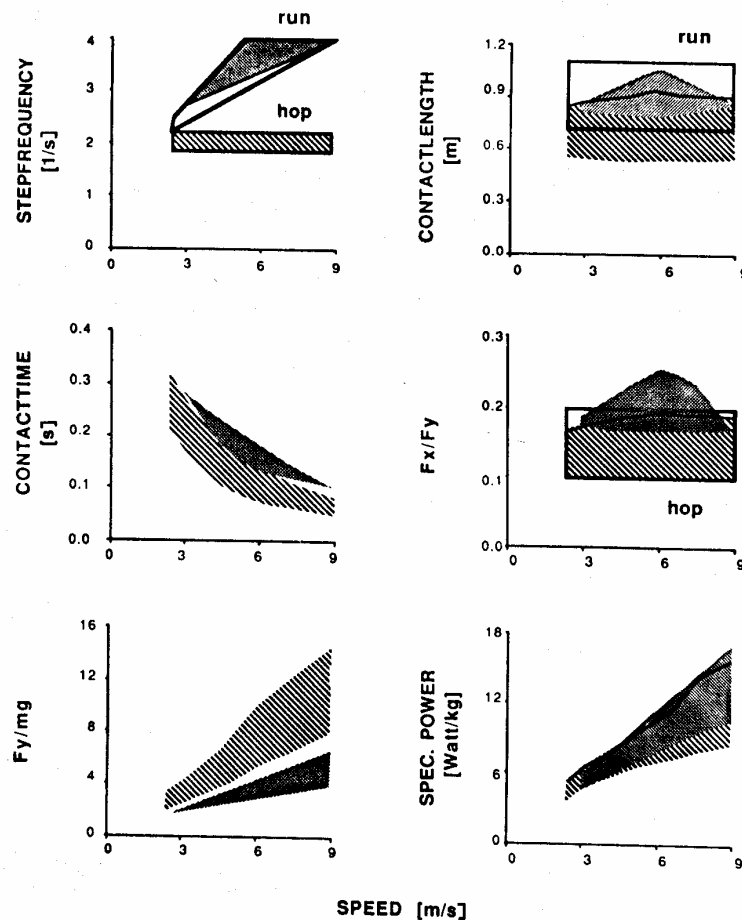
Fig. 8. Running (stippled) versus hopping (hatched). The thin line marks the upper boundary of the hatched area. By assigning values to two parameters (bold outlined) in addition to body mass (64 kg) and leg length (0.9 m), the other parameters can be calculated. [Selected regions—running: step frequency $= (0.31 \pm 0.05)$ speed $+ (1.8 \pm 0.2)$ s^{-1} (speed in m s^{-1}), and 0.7 m < contact length < 1.1 m; hopping: 1.8 s^{-1} < hopping frequency < 2.2 s^{-1}, and 0.1 < F_x/F_y < 0.2. As shown in Fig. 7, this is equivalent to a restriction of contact length. Due to the mutual interference of the restrictions, the shaded regions where solutions were found do not cover the regions of the assignments.]

Hopping. Quadrupedal animals increase their stride frequency during trotting until they attain the galloping frequency which is nearly constant across the entire speed range and is similar for animals of comparable mass (Heglund and Taylor, 1988). A human subject selects the same frequency for hopping forward. In the following, this frequency, as well as the ratio of horizontal to vertical force (0.1–0.3), is used to set the point of operation of the spring-mass system during simulation of human hopping.

The hopping frequency is similar to the maximum stride frequency attained at maximum running speed. However, for each stride during running there are two steps or two bounces. Thus, the hopping frequency is lower than the step frequency selected during running. Recalling the coupling between vertical displacement and frequency, then hopping results in a bumpier ride than running. The power fluctuations of the center of mass per stride do not differ much from running. They seem to be lower at high speed, resulting in a slightly lower cost of transport. All parameters including con-

tact time and ground reaction force remain within physiologically reasonable boundaries.

CONCLUSION

Investigating the performance of a simple spring-mass model has resulted in several general insights. A simple model describes human running and hopping remarkably well. Dimensionless parameters such as the specific ground reaction force and specific landing velocity are suitable to specify hopping patterns. Physiological limits largely defined by muscle properties and skeletal dimensions result in a narrow frequency band where hopping is possible. At the preferred frequency high hops can be produced economically.

In forward or planar hopping the specific spring length must be included as an additional parameter determining the performance of the spring-mass system. Not only does the stiffness of the spring and the loading during landing impact determine the ground

reaction force, but also the centrifugal force contributes due to rotation of the mass around the foothold. Expecially for the low angles of landing velocity selected by average runners or hoppers (5–25°) the dependency of hopping frequency on this angle is steep. The advantage of selecting such a flat landing angle is a maximum contact length with low contact forces. Besides predicting useful relationships between various parameters the model allows correct calculation of the mechanical energetics of the center of mass based on only a few kinematic and/or dynamic parameters.

Clearly the validity of such a simple model should not be overstressed. For example, animals in general do not have a similar take off and landing velocity as assumed in the model. They take off with straightened legs and land with bent legs. Muscles and tendons control the compliance during landing, avoiding axial impact loads of the bones. The distance of deceleration during landing is shorter than of acceleration during take off, i.e. the leg has to be more stiff during landing than during take off. By implementing different stiffnesses during landing and take off it would be possible to investigate the influence of deviations from the symmetrical model. However, such a correction would not lead to major changes in the general performance of the model. The heelstrike in human running can be considered as a mechanism to deliver part of the additional energy necessary for deceleration during landing. It is largely due to deceleration of distal leg masses (Alexander et al., 1986). The introduction of a leg mass also requires a joint between body and leg in order to be able to conserve torque during a step cycle, thus adding two independent parameters to the system. Such a model served as a mathematical basis for hopping machines which largely mimic animal locomotion (Raibert, 1984, 1985).

The fact that the spring-mass model is successful in predicting and describing general features of animal locomotion does not depend on a detailed agreement of the real leg with the assumed linear, massless spring. General features like the conservation of momentum during ground contact, coupling of horizontal traveling time to the elastic rebound of the spring, coupling between horizontal and vertical forces etc., are the basic conditions making this model successful. It should be stressed that the model does not imply that all hopping and running in animals is just elastic bouncing with some deviations. It states that, even in the case of actively supplied forces, a bouncing system behaves very similarly to a spring-mass system.

REFERENCES

Alexander, R. McN. and Vernon, A. (1975) The mechanics of hopping by kangaroos (Macropodidae). *J. Zool. (Lond.)* **177**, 265–303.

Alexander, R. McN., Bennett, M. B. and Ker, R. F. (1986) Mechanical properties and functions of the paw pads of some mammals. *J. Zool. A.* **104**, 405–419.

Blickhan, R. (1989) Running and hopping. *Energy Transformations in Cells and Organisms* (Edited by Wieser, W.). Georg Thieme, Stuttgart.

Blickhan, R. and Full, R. J. (1987) Energetics of locomotion in the ghost crab *Ocypoda quadrata*. II. Mechanics of the center of mass. *J. exp. Biol.* **130**, 155–174.

Blickhan, R., Farley, C. T. and Taylor, C. R. (1986) What determines hopping galloping stride frequency? *Proc Int. Union Physiol. Sci.* **XIV**, 14.

Cavagna, G. A., Thys, H. and Zamboni, A. (1976) The sources of external work in level walking and running. *J. Physiol. (Lond.)* **262**, 639–657.

Cavagna, G. A., Heglund, N. C. and Taylor, C. R. (1977) Mechanical work in terrestrial locomotion: two basic mechanisms of minimizing energy expenditure. *Am. J. Physiol.* **233**, R243–R261.

Close, R. I. (1972) Dynamic properties of mammalian skeletal muscles. *Physiol. Rev.* **52**, 129–197.

Farley, C. T., Blickhan, R. and Taylor, C. R. (1985) Mechanics of human hopping: model and experiments. *Am. Zool.* **25**, 54A.

Heglund, N. C. and Taylor, C. R. (1988) Speed, stride frequency and energy cost per stride: how do they change with size and gait? *J. exp. Biol* **138**, 131–318.

Heglund, N. C., Taylor, C. R. and McMahon, T. A. (1974) Scaling stride frequency and gait to animal size: mice to horses. *Science* **186**, 1112–1113.

Heglund, N. C., Cavagna, G. A. and Taylor, C. R. (1982) Energetics and mechanics of terrestrial locomotion. III. Energy changes of the center of mass as a function of speed and body size in birds and mammals. *J. exp. Biol.* **79**, 41–56.

McMahon, T. A., Valiant, G. and Frederick, E. C. (1986) Croucho running. *J. appl. Physiol.* **62**, 2326–2337.

Raibert, M. H. (1984) Hopping in legged systems—Modelling and simulation for the 2D one-legged case. *IEEE Trans. Syst. Man. Cybern.* **14**, 451–463.

Raibert, M. H. (1985) *Legged Robots That Balance*. MIT Press, Cambridge, MA.

Taylor, C. R. (1985) Force development during sustained locomotion: a determinant of gait, speed and metabolic power. *J. exp. Biol.* **115**, 253–262.